PROBLEM 1

Problem statement: Consider a T2D2 element. The element is loaded by a centrifugal force of $b_0 x$. The nodal coordinates are $x_1^e = 0.0$mm and $x_2^e = 3.0$ mm. Determine:
(1) The element body force vector using analytical integration
(2) The element body force vector using Gauss integration
Provide details of the computation and not just the final answers.
Start from: $f_\Omega^e = \int_{x_1^e}^{x_2^e} (N^e)^T b(x)\, dx$

Solution:
(1) analytical integration:

$$N_1^e = \frac{x_2^e - x}{l^e} = \frac{3-x}{3} = 1 - 1/3\,x$$

$$N_2^e = \frac{x - x_1^e}{l^e} = \frac{x-0}{3} = 1/3\,x$$

$$f_\Omega^e = \int_{x_1^e}^{x_2^e} (N^e)^T b(x)\, dx = \int_0^3 \begin{bmatrix} 1 - 1/3\,x \\ 1/3\,x \end{bmatrix} b_0 x\, dx$$

↳ break into two integrations:

$$b_0 \int_0^3 (x - \tfrac{1}{3}x^2)\, dx = b_0 \left(\tfrac{1}{2}x^2 - \tfrac{1}{9}x^3\right)\Big|_0^3 = b_0 \left(\tfrac{1}{2}9 - \tfrac{1}{9}27\right) = b_0 \cdot \tfrac{3}{2}$$

$$b_0 \int_0^3 (1/3\,x^2)\, dx = b_0 \left(\tfrac{1}{9}x^3\right)\Big|_0^3 = 3b_0$$

$$\boxed{f_\Omega^e = \begin{bmatrix} 3b_0/2 \\ 3b_0 \end{bmatrix}}$$

(2) gauss integration

$$f_\Omega^e = J\left(f(z_1)\cdot w_1 + f(z_2)\cdot w_2\right)$$

$$x = \frac{b-a}{2} z + \frac{b+a}{2} = \frac{3}{2} z + \frac{3}{2} \qquad J = \frac{3}{2}$$

$z_1 = -\sqrt{1/3},\ w_1 = 1$
$z_2 = +\sqrt{1/3},\ w_2 = 1$

first term:

$$f(z_1) = \frac{3 - \left(\frac{3}{2}(-\sqrt{1/3}) + \frac{3}{2}\right)}{3} = 1 + \frac{1}{2\sqrt{3}} - \frac{1}{2} = \frac{1}{2} + \frac{1}{2\sqrt{3}}$$

$$f(z_2) = \frac{3 - \left(\frac{3}{2}(\sqrt{1/3}) + \frac{3}{2}\right)}{3} = 1 - \frac{1}{2\sqrt{3}} - \frac{1}{2} = \frac{1}{2} - \frac{1}{2\sqrt{3}}$$

$$f_\Omega^e = \frac{3}{2}\left(\left(\tfrac{1}{2} + \tfrac{1}{2\sqrt{3}}\right)\cdot 1 + \left(\tfrac{1}{2} - \tfrac{1}{2\sqrt{3}}\right)\cdot 1\right) = \frac{3}{2}(1) = 3/2$$

second term:

$$f(z_1) = \frac{\frac{3}{2}(-\sqrt{1/3}) + \frac{3}{2}}{3} = -\frac{1}{2\sqrt{3}} + 1$$

$$f(z_2) = \frac{\frac{3}{2}(+\sqrt{1/3}) + \frac{3}{2}}{3} = \frac{1}{2\sqrt{3}} + 1$$

$$f_\Omega^e = \frac{3}{2}\left(\left(1 - \tfrac{1}{2\sqrt{3}}\right)\cdot 1 + \left(1 + \tfrac{1}{2\sqrt{3}}\right)\cdot 1\right) = \frac{3}{2}(2) = 3$$

$$f_\Omega^e = \begin{bmatrix} \frac{3}{2}b_0 \\ 3b_0 \end{bmatrix} \left\{ \text{same as analytical} \right.$$

PROBLEM 2:

Problem statement: Consider a T2D3 element. The nodal coordinates are $x_1^e = 1.0$ mm, $x_2^e = 3.0$ mm, and $x_3^e = 5.0$ mm. Let $AE = 1$ N. Determine:

(1) The numerical value of K_{23}^e of the stiffness matrix using analytical integration

(2) The numerical value of K_{23}^e of the stiffness matrix using gauss integration

Provide details of the computation and not just the final answers. Start from: $K^e = \int_{x_1^e}^{x_3^e} (B^e)^T EA (B^e)\, dx$

Solution:

(1) value of K_{23}^e using analytical integration:

$$K^e = \int_{x_1^e}^{x_3^e} (B^e)^T EA (B^e)\, dx$$

$$f(x)dx = \frac{1}{J^e} B^e(\xi)\, dx \qquad dx = J^e d\xi$$

$$K^e(\xi) = \int_{-1}^{+1} \frac{1}{J^e} [B^e(\xi)]^T EA [B^e(\xi)] \cdot \frac{1}{J^e} \cdot J^e\, d\xi$$

for a T2D3 element: $B_2^e = -2\xi$, $B_3^e = \xi + 1/2$

$$K_{23}^e = \int_{-1}^{+1} \frac{EA}{J} B_2^e B_3^e\, d\xi = \frac{2EA}{l^e} \int_{-1}^{+1} (-2\xi)(\xi + 1/2)\, d\xi \quad \text{(will cancel)}$$

$$= \frac{2EA}{l^e} \int_{-1}^{+1} (-2\xi^2 - \xi)\, d\xi = \frac{2EA}{l^e} \cdot \left[-\frac{2}{3}\xi^3 - \frac{1}{2}\xi^2 \right] \Big|_{-1}^{+1}$$

$$= \frac{2EA}{l^e} \left[-\frac{2}{3}(1)^3 - \frac{2}{3}(-1)^3 \right] = \frac{2EA}{l^e} \cdot \frac{-4}{3} = -16 \frac{EA}{6 l^e}$$

$$K_{23}^e = -16 \cdot \frac{EA}{6 l^e} = -16 \frac{(1)}{6(5-1)} = \boxed{-0.666 \text{ N/mm}}$$

(2) value of K_{23}^e using gauss integration:

$$K_{23}^e = \frac{2EA}{l^e} \int_{-1}^{+1} (-2\xi^2 - \xi)\, d\xi \rightarrow P = 2;\ n_g = 2$$

$\xi_1 = -\sqrt{1/3}$, $w_1 = 1$

$\xi_2 = +\sqrt{1/3}$, $w_2 = 1$

$$\int_{-1}^{+1} f(\xi)\, d\xi = w_1 \cdot f(\xi_1) + w_2 \cdot f(\xi_2)$$

$$f(\xi_1) = -2(-\sqrt{1/3})^2 - (-\sqrt{1/3}) = -2(1/3) + \sqrt{1/3} = -2/3$$

$$f(\xi_2) = -2(\sqrt{1/3})^2 - (\sqrt{1/3}) = -2(1/3) - \sqrt{1/3} = -2/3$$

$$\int_{-1}^{+1} f(\xi)\, d\xi = 1 \cdot (-2/3) + 1 \cdot (-2/3) = -4/3$$

$$-\frac{4}{3} \cdot \frac{2EA}{l^e} = -16 \cdot \frac{EA}{6 l^e} \quad \left\{ \text{same as analytical} \right.$$

$$K_{23}^e = -16 \frac{EA}{6 l^e} = -16 \frac{(1)}{6(4)} = \boxed{-0.666 \text{ N/mm}}$$